ASYMPTOTIC ANALYSIS OF THE CHARACTERISTIC POLYNOMIAL FOR THE ELLIPTIC GINIBRE ENSEMBLE.

Combinatoire Elliptique et au delà,
Institut d'Études Scientifiques de Cargèse.

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Quentin FRANCOIS (CEREMADE, Université Dauphine PSL),
Joint work with David GARCIA-ZELADA (LPSM, Sorbonne Université).
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Outline

Elliptic Ginibre Ensemble

Definition (Girko Matrix)

$$A_n = (a_{i,j}, 1 \leq i, j \leq n) \text{ i.i.d}$$

I.i.d Matrices

Definition (Girko Matrix)

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Definition (Ginibre Ensemble)

$$A_n \sim \mathrm{d}\mathbb{P}_n(A) = \frac{1}{Z_n} \exp\left(-Tr[AA^*]\right)\mathrm{d}A.$$

$$a_{i,j} \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ i.i.d} \implies A_n \sim \mathit{Ginibre}.$$

Hermitian Matrices

Definition (Wigner matrix)

$$A_n = (a_{i,j}, 1 \leq i,j \leq n), \quad (a_{i,j}, i < j) \ i.i.d, \quad a_{i,j} = \overline{a_{j,i}}.$$

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Definition (Gaussian Unitary Ensemble)

$$A_n \sim \mathrm{d}\mathbb{P}[A] = \frac{1}{Z_n} \exp\left(-\frac{1}{2}\mathrm{T}r[A^2]\right)\mathrm{d}A \text{ on } \mathcal{H}_n.$$

$$a_{i,j} \underset{i<j}{\sim} \mathcal{N}_{\mathbb{C}}(0,1), a_{i,i} \sim \mathcal{N}(0,1) \implies A_n \sim \text{GUE}.$$

The Elliptic Ginibre Ensemble at $t \in [0,1]$

$$A_{n,t} \sim \mathrm{d}\mathbb{P}_{n,t}(A) = \frac{1}{Z_{n,t}} \exp\left(-\frac{1}{1-t^2}\mathrm{Tr}\left[A^*A - \frac{t}{2}(A^2 + (A^*)^2)\right]\right)\mathrm{d}A.$$

$$A_n^{(1)}, A_n^{(2)} \text{ indep. GUE} \implies A_{n,t} = \sqrt{\frac{1+t}{2}}A_n^{(1)} + i\sqrt{\frac{1-t}{2}}A_n^{(2)} \sim \mathrm{EGE}_t$$

- $t = 0$: $A_{n,0} \sim$ Ginibre.
- $t = 1$: $A_{n,1} \sim$ GUE.
- $t \in [0,1]$: $A_{n,t} \sim \mathrm{EGE}_t$, $(a_{i,j}, i < j)$ $i.i.d$, $\mathbb{E}[a_{1,2}a_{2,1}] = t$.

Eigenvalue measure

Definition (Eigenvalue measure)

$$\mu_n(A) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(A)}.$$

Convergence of random measures

- (Classical convergence) $\mu_n \implies \mu$ if $\forall f \in \mathcal{C}_b : \int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$.
- $\mu_n \underset{\mathbb{P}}{\implies} \mu$ if $\forall f \in \mathcal{C}_b : \int f \mathrm{d}\mu_n \underset{\mathbb{P}}{\to} \int f \mathrm{d}\mu$.
- $\mu_n \underset{a.s}{\implies} \mu$ if $a.s, \forall f \in \mathcal{C}_b : \int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$.

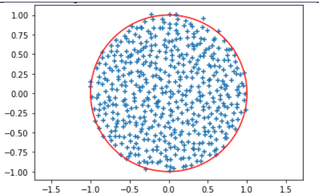
Eigenvalue measure convergence

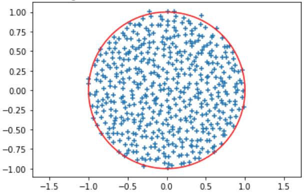
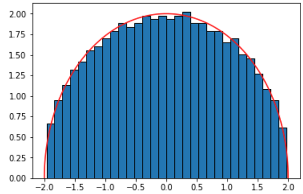
$t = 0$: Ginibre $\subset$ Girko

$$\mathbb{E}[|a_{1,1}|^2] = 1$$

$$\mu\left(\frac{1}{\sqrt{n}}A_n\right) \underset{a.s}{\Longrightarrow} \mu = \frac{1}{\pi}\mathbb{1}_{|z|<1}\mathrm{d}z$$

$$E_0 = \{z : |z| < 1\}.$$



$t = 0$: Ginibre $\subset$ Girko $\mathbb{E}[\|a_{1,1}\|^2] = 1$ $\mu\left(\frac{1}{\sqrt{n}}A_n\right) \underset{a.s}{\Longrightarrow} \mu = \frac{1}{\pi}\mathbf{1}_{\|z\|<1}\mathrm{d}z$ $E_0 = \{z : \|z\| < 1\}$.	
$t = 1$: GUE $\subset$ Wigner $\mathbb{E}[\|a_{1,2}\|^2] = 1$ $\mu\left(\frac{1}{\sqrt{n}}A_n\right) \underset{a.s}{\Longrightarrow} \mu = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{\|x\|<2}\mathrm{d}x$ $E_1 = [-2, 2]$	

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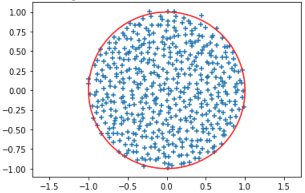
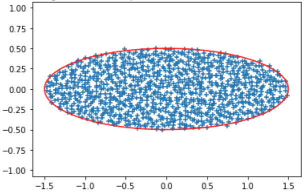


$t \in (0, 1)$: Elliptic

$$\mathbb{E}[a_{1,2} a_{2,1}] = t$$

$$\mu\left(\frac{1}{\sqrt{n}} A_{n,t}\right) \underset{a.s}{\Longrightarrow} \mu_t = \frac{1}{\pi(1-t^2)} \mathbf{1}_{z \in E_t} \mathrm{d}z$$

$$E_t = \left\{ z : \left(\frac{\Re(z)}{1+t}\right)^2 + \left(\frac{\Im(z)}{1-t}\right)^2 \leq 1 \right\}$$

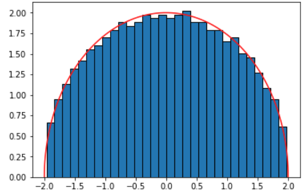


$t = 1$: GUE $\subset$ Wigner

$$\mathbb{E}[|a_{1,2}|^2] = 1$$

$$\mu\left(\frac{1}{\sqrt{n}} A_n\right) \underset{a.s}{\Longrightarrow} \mu = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{|x|<2} \mathrm{d}x$$

$$E_1 = [-2, 2]$$



Characteristic polynomial asymptotics

Definition (reciprocal characteristic polynomial)

$A_n \sim$ Girko, $q_n \in \mathcal{H}(\mathbb{D})$:

$$\mathbb{D} \ni z \mapsto q_n(z) = \det\left(1 - \frac{zA_n}{\sqrt{n}}\right) = z^n p_n\left(\frac{1}{z}\right).$$

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Theorem [BCG22]

$$q_n \underset{law}{\to} f_\infty \in \mathcal{H}(\mathbb{D}) : z \mapsto \sqrt{1 - z^2\mathbb{E}[a_{11}^2]}\exp\left\{-\sum_{k\geq 1}\frac{X_k z^k}{\sqrt{k}}\right\},$$

with $\{X_k\}_{k\geq 1}$ independent complex centered Gaussians:

$$\mathbb{E}[X_k^2] = \mathbb{E}[a_{11}^2]^k \quad \text{and} \quad \mathbb{E}|X_k|^2 = 1.$$

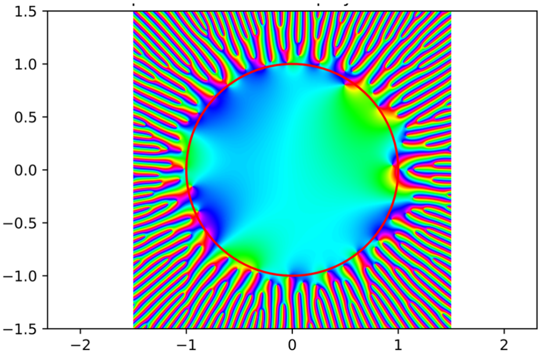


Figure 1: Phase portrait of q_n for $A_n \sim$ Ginibre, $n = 500$.

Reciprocal characteristic polynomial

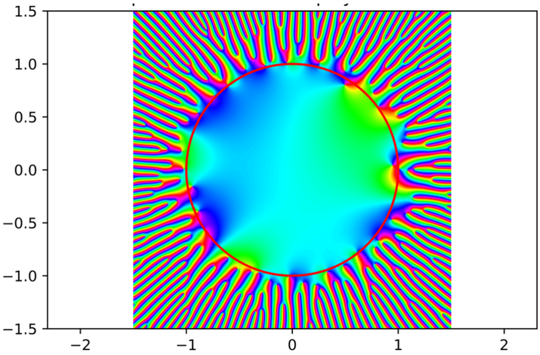


Figure 1: Phase portrait of q_n for $A_n \sim$ Ginibre, $n = 500$.

Corollary (Convergence of the spectral radius)

$$\rho_n = \max\left\{\frac{1}{\sqrt{n}}|\lambda_i|, 1 \leq i \leq n\right\} \xrightarrow{\mathbb{P}} 1.$$

Main Result

Normalized characteristic polynomial

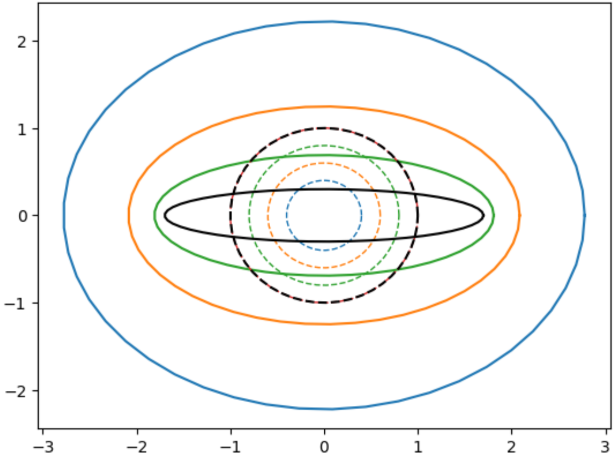
$A_{n,t} \sim \mathrm{EGE}_t$, $f_{n,t} \in \mathcal{H}(\mathbb{D})$.

$$f_{n,t} : \mathbb{D} \ni z \mapsto z^n \det\left(\frac{1}{z} + tz - \frac{1}{\sqrt{n}} A_{n,t}\right) \exp\left(-\frac{ntz^2}{2}\right).$$

Intuition : $g_t : z \mapsto \frac{1}{z} + tz$ maps $\mathbb{D}$ to $\mathbb{C} \setminus \mathcal{E}_t$.

Mapping circles to ellipses

$$g_t(D_r) = \left\{ z : \left(\frac{\Re(z)}{\frac{1}{r} + tr} \right)^2 + \left(\frac{\Im(z)}{\frac{1}{r} - tr} \right)^2 \leq 1 \right\}$$



Theorem

$$\forall t \in [0,1] : f_{n,t} \underset{law}{\to} f_t \in \mathcal{H}(\mathbb{D}) : z \mapsto \kappa_t(z) \exp\left(- \sum_{k \geq 1} X_k^{(t)} \frac{z^k}{\sqrt{k}} \right),$$

$\{X_k^{(t)}\}_{k\geq 1}$ independent Gaussians,

$$\mathbb{E}\left(X_k^{(t)}\right)^2 = t^k, \quad \mathbb{E}|X_k^{(t)}|^2 = 1,$$

$$\kappa_t(z) = \exp\left(-\frac{1}{2} \sum_{k\geq 1} \frac{h_{k,t}}{k} z^{2k} \right) \cdot \exp\left(\frac{tz^2}{2(1-tz^2)} \right).$$

Illustration of the Theorem

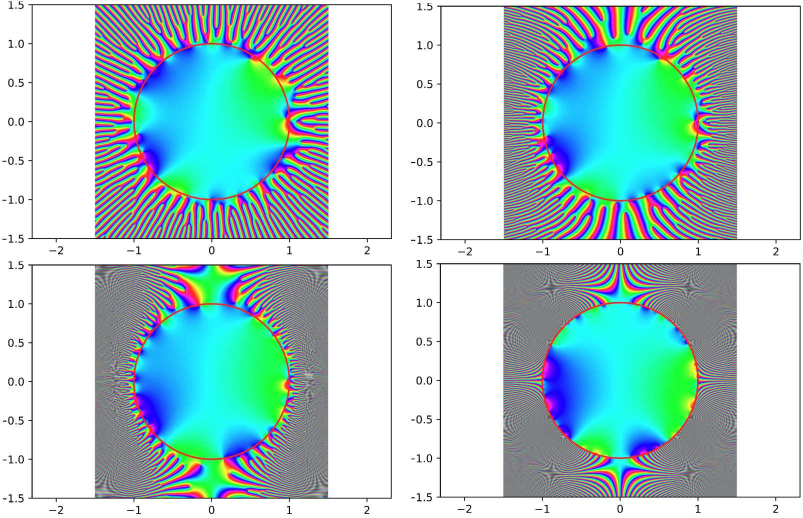


Figure 3: Phase portrait of $f_{n,t}$ for $n = 250$ and $t = 0$ (top left), 0.3 (top right), 0.6 (bottom left) and 1 (bottom right).

Proof outline

$$f_n(z) = \sum_{k \geq 0} \xi_k^{(n)} z^k, \quad \xi_k^{(n)} \text{ random.}$$

(a) $\{f_n\}_{n \geq 1}$ **tight**.

(b) $\forall m \geq 0 : (\xi_0^{(n)}, \dots, \xi_m^{(n)}) \underset{law}{\rightarrow} (\xi_0, \dots, \xi_m)$.

$$\Longrightarrow \; f_n \underset{law}{\rightarrow} f : z \mapsto \sum_{k \geq 0} \xi_k z^k.$$

Tightness

Proof of Tightness

Lemma (Uniform control)

$$(||f_{n,t}||_K)_{n\geq 1} \text{ tight } \implies \{f_{n,t}\}_{n\geq 1} \text{ tight.}$$

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$$\text{Bounded second moment} \implies \text{tightness.}$$

$$\left(\sup_{z\in K} \mathbb{E}[|f_{n,t}(z)|^2]\right)_{n\geq 1} \text{ bounded} \underset{|f(z)|^2 \text{ subhar}}{\iff} \left(\mathbb{E}[||f_{n,t}||_K^2]\right)_{n\geq 1} \text{ bounded}$$

$$\Downarrow$$

$$(||f_{n,t}||_K)_{n\geq 1} \text{ tight}$$

Expression of $\mathbb{E}|f_{n,t}(z)|^2$ via orthogonal polynomials

$$A_{n,t} \sim \frac{1}{Z_{n,t}} \exp\left(-V_t(A)\right) \mathrm{d}A,$$

$$V_t(z) = \frac{1}{1-t^2}\left(|z|^2 - \frac{t}{2}(z^2 + \overline{z}^2)\right) = \frac{\Re(z)^2}{1+t} + \frac{\Im(z)^2}{1-t}.$$

Orthogonal polynomials for V_t: **Hermite polynomials**

$$\int_{\mathbb{C}} H_n\left(\frac{z}{\sqrt{t}}\right) H_m\left(\frac{\overline{z}}{\sqrt{t}}\right) V_t(z) dz = n! \left(\frac{1}{t}\right)^n \delta_{n=m}.$$

Hermite expression

Lemma (Hermite expression)

$\forall z \in \mathbb{D} \setminus \{0\}$:

$$\mathbb{E}\left[|f_{n,t}(z)|^2\right] = |z|^{2n} \frac{n!}{n^n} \sum_{k=0}^{n} \frac{t^k}{k!} \left| H_k \left(\sqrt{\frac{n}{t}} (z^{-1} + tz) \right) \right|^2 |e^{-ntz^2}|.$$

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Lemma (Second moment convergence, using [ADM23])

$$\mathbb{E}\left[|f_{n,t}|^2\right] \to \mathcal{F}_t : \mathbb{D} \setminus \{0\} \to (0, \infty).$$

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$$\mathbb{E}[||f_{n,t}||^2_{D_r}] \leq C_r \implies \text{tightness}.$$

Limit function

Coefficients convergence

$$f_{n,t}(z) = \det\left(1 + tz^2 - \frac{A_{n,t}}{\sqrt{n}}\right)\exp\left(-nt\frac{z^2}{2}\right) = \sum_{k\geq 0}\xi_{k,t}^{(n)}z^k.$$

Objective

$$\forall m \geq 0 : (\xi_{0,t}^{(n)}, \ldots, \xi_{m,t}^{(n)}) \underset{law}{\rightarrow} (\xi_{0,t}, \ldots, \xi_{m,t}).$$

Modified Chebyshev polynomials

$$P_{k+1}^{(t)} = XP_k^{(t)} - tP_{k-1}^{(t)}, \quad P_0^{(t)} = 2, \quad P_1^{(t)} = X$$

verify:

$$\sum_{k\geq 1} P_k^{(t)}(w)\frac{z^k}{k} = \log\left(\frac{1}{1 - wz + tz^2}\right).$$

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$$\begin{aligned} f_{n,t}(z) &= \det\left(1 - z\frac{A_{n,t}}{\sqrt{n}} + tz^2\right)\exp\left(-nt\frac{z^2}{2}\right) \\ &= \exp\left(-\sum_{k\geq 1} Tr\left[P_k^{(t)}\left(\frac{A_{n,t}}{\sqrt{n}}\right)\right]\frac{z^k}{k}\right)\exp\left(-nt\frac{z^2}{2}\right). \end{aligned}$$

Trace convergence

Goal

$$\left(Tr\left[P_k^{(t)}\left(\frac{A_{n,t}}{\sqrt{n}} \right) \right] + nt\delta_{k=2} \right)_{k\geq 1} \to \text{ Gaussian family.}$$

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Expectation convergence

$$\forall k \geq 1, t \in [0,1] : \mathbb{E}\left[Tr\left[P_k^{(t)}\left(\frac{A_{n,t}}{\sqrt{n}} \right) \right] \right] + nt\delta_{k=2} \underset{n\to\infty}{\longrightarrow} e_{k,t}.$$

Gaussian fluctuations

$$X_{k,t}^{(n)} = Tr\left[P_k^{(t)}\left(\frac{A_{n,t}}{\sqrt{n}}\right)\right] - \mathbb{E}\left[Tr\left[P_k^{(t)}\left(\frac{A_{n,t}}{\sqrt{n}}\right)\right]\right].$$

Convergence to Gaussian family $\iff$ Pairing.

Isserlis-Wick theorem

$\{Z_k\}_k$ Gaussian $\iff \forall i_1, \ldots, i_n,\ s_1, \ldots, s_n$:

$$\mathbb{E}[Z_{i_1}^{(s_1)} \ldots Z_{i_n}^{(s_n)}] = \sum_{\pi \in \mathcal{P}_2(n)} \prod_{(k,l)\in\pi} \mathbb{E}[Z_{i_k}^{(s_k)} Z_{i_l}^{(s_l)}].$$

Lemma (Monomial fluctuations)

$$\left(n^{-k/2}\left(\mathrm{Tr}[A_{n,t}^k] - \mathbb{E}[\mathrm{Tr}[A_{n,t}^k]]\right)\right)_{k\geq 0} \to \text{ Gaussian family.}$$

Trace expansion

$$Tr[A^k] = \sum_{i=(i_1,\ldots,i_k)} a_{i_1,i_2} a_{i_2,i_3} \ldots a_{i_{k-1},i_k} a_{i_k,i_1} = \sum_{\psi:[k]\to[n]} a_\psi.$$

$$Tr[A^k] = \sum_{i=(i_1,\dots,i_k)} a_{i_1,i_2} a_{i_2,i_3} \dots a_{i_{k-1},i_k} a_{i_k,i_1} = \sum_{\psi:[k]\to[n]} a_\psi.$$

Fix $m \geq 1$, $k_1, \dots, k_m \geq 1$ and $s_1, \dots, s_m \in \{0, 1\}$.

$$\begin{aligned} M^{(n)}_{\mathbf{k},t} &= \mathbb{E}\left[\prod_{j=1}^{m} n^{-\frac{k_j}{2}} (Tr(A^{k_j}) - \mathbb{E}[Tr(A^{k_j})])^{(s_j)}\right] \\ &= n^{-\frac{k}{2}} \sum_{\psi_1,\dots,\psi_m} \mathbb{E}\left[\prod_{j=1}^{m} (a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}\right]. \end{aligned}$$

Graph representation

$$\psi_j : [k_j] \to [n] \Longrightarrow G_j = (V_j, E_j)$$
$$V_j = Im(\psi_j),\ \ E_j = \{(\psi_j(l), \psi_j(l+1)), 1 \leq l \leq k_j\}$$

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Ex: $k_1 = 9, k_2 = 5,\ \psi_1 = [1, 5, 3, 5, 2, 4, 2, 6, 2], \psi_2 = [2, 8, 6, 8, 1]$.

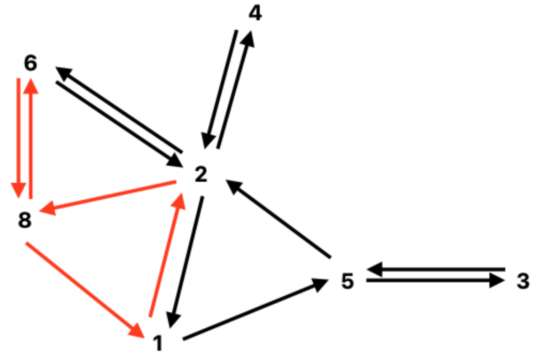

Figure 4: Graph associated to ψ_1(black) and ψ_2 (red).

Grouping graphs

$$\psi : [k] \to [n] \implies \ker\psi \in \mathcal{P}(k) : u \overset{\ker\psi}{\sim} v \iff \psi(u) = \psi(v).$$

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Lemma (group by partition)

$$\ker \psi_{\mathbf{k}} = \ker \varphi_{\mathbf{k}} \implies \mathbb{E}\left[\prod_{j=1}^{m} (a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}\right] = \mathbb{E}\left[\prod_{j=1}^{m} (a_{\varphi_j} - \mathbb{E}[a_{\varphi_j}])^{(s_j)}\right].$$

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$$M_{\mathbf{k},t}^{(n)} = n^{-\frac{k}{2}} \sum_{\pi \in \mathcal{P}(k)} c_k^{(n)}(\pi) a_{k,t}(\pi),$$

$$a_{k,t}(\pi) = \mathbb{E}\left[\prod_{j=1}^{m}(a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}\right] \text{ for } \ker\psi_{\mathbf{k}} = \pi,$$

$$c^{(n)}(\pi) = \#\{\psi : [k] \to [n] \mid \ker\psi = \pi\} = n(n-1)\ldots(n - \#\pi + 1).$$

Definition (Bar graph)

$$G = (E, V) \Longrightarrow \overline{G} = (\overline{E}, V),$$

$$\text{G connected: } q_1(G) = |\overline{E}| - \frac{|E|}{2}, \quad q_2(G) = |V| - |\overline{E}|.$$

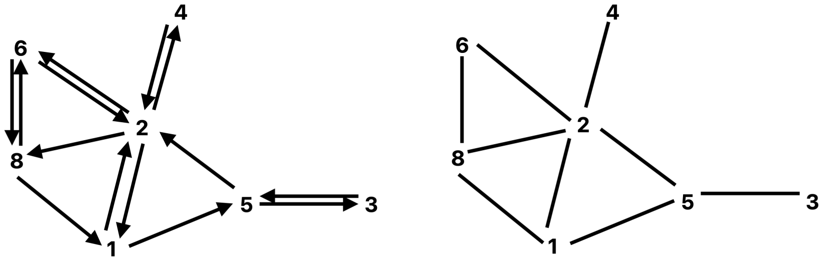

Figure 5: The map $G \Longrightarrow \overline{G}$. Here $q_1(G) = 2, q_2(G) = -2$.

Asymptotic expansion

$$M^{(n)}_{\mathbf{k},t} = \sum_{\pi \in \mathcal{P}(k)} n_{\#(\pi)} n^{-\frac{k}{2}} a_{k,t}(\pi).$$

$$\pi \underset{\ker^{-1}}{\longrightarrow} \psi \longrightarrow G_\psi = \cup_{i \in I} \Gamma_i, \ (q_{i,1}, q_{i,2}) = (q_1(\Gamma_i), q_2(\Gamma_i)).$$

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Leading terms

$$\#\pi - \frac{k}{2} = |V_G| - \frac{|E_G|}{2} = \sum_{i \in I} (q_{i,1} + q_{i,2}) =: q(\pi).$$

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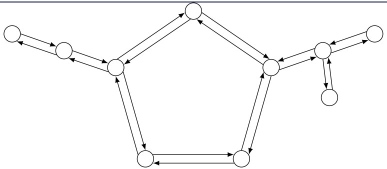
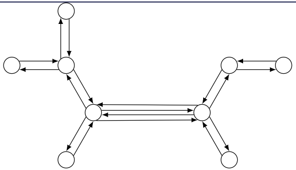
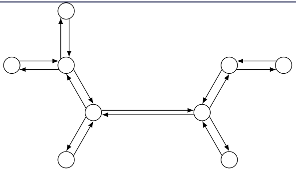
$$\pi \underset{\ker^{-1}}{\longrightarrow} \psi \longrightarrow G_\psi = \cup_{i \in I} \Gamma_i, \; (q_{i,1}, q_{i,2}) = (q_1(\Gamma_i), q_2(\Gamma_i)).$$

Leading terms

$$\#\pi - \frac{k}{2} = |V_G| - \frac{|E_G|}{2} = \sum_{i \in I} (q_{i,1} + q_{i,2}) =: q(\pi).$$

Which graphs are q-positive ?

Double Unicyclic		$q_1 = 0$ $q_2 = 0$
2-4 Tree		$q_1 = -1$ $q_2 = 1$
Double tree		$q_1 = 0$ $q_2 = 1$

Conditions for non-zero contributions

$$M_{\mathbf{k},t}^{(n)} = n^{-\frac{k}{2}} \sum_{\psi_1,\dots\psi_m} \mathbb{E}\left[\prod_{j=1}^{m} (a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}\right] = \sum_{\pi \in \mathcal{P}(k)} n_{\#(\pi)} n^{-\frac{k}{2}} a_{k,t}(\pi).$$

1. No simple edge in Γ_i.
2. $q_{i,1} \leq 0$, $q_{i,2} \leq 1$ ($=$ iff $\overline{G}$ tree).
3. $\forall i, j : G_j \neq \Gamma_i$.
4. $q_{i,1} + q_{i,2} \leq 0$.
5. $q_{i,1} + q_{i,2} = 0 \iff \Gamma_i'$s are DU or 2-4 trees.

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Proposition (Asymptotic contributions)

$$M_{\mathbf{k},t}^{(n)} = \sum_{\pi\in DU\&FT} a_{k,t}(\pi) + o(1)$$

- Double unicyclic $\equiv$ pairing two unicyclic graphs.
- 2-4 trees $\equiv$ pairing two doubles trees.

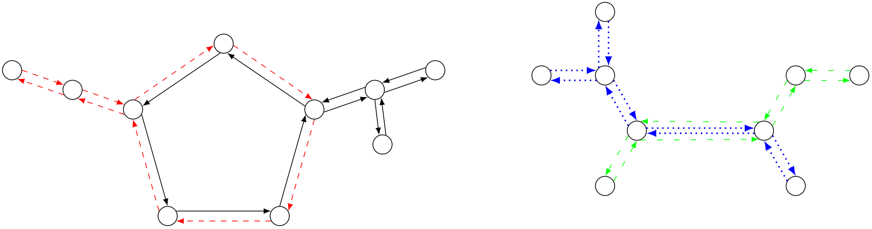

Figure 6: Blue and red (resp.) blue and green are paired.

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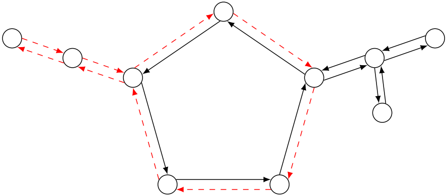

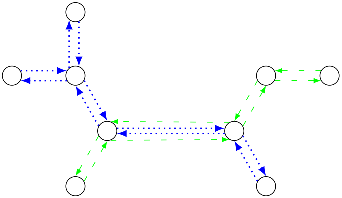

Figure 6: Blue and red (resp.) blue and green are paired.

$$\left(n^{-k/2}\left(\mathrm{Tr}[A_{n,t}^k] - \mathbb{E}[\mathrm{Tr}[A_{n,t}^k]]\right)\right)_{k\geq 0} \longrightarrow (Z_{k,t})_{k\geq 0} \text{ Gaussian.}$$

Example computation of $a_t(1, 2)$ for DU^{opp}

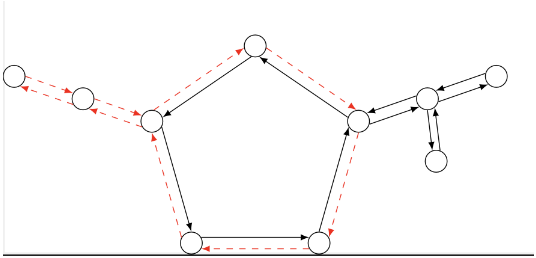

Figure 7: Double (opposite) unicyclic component.

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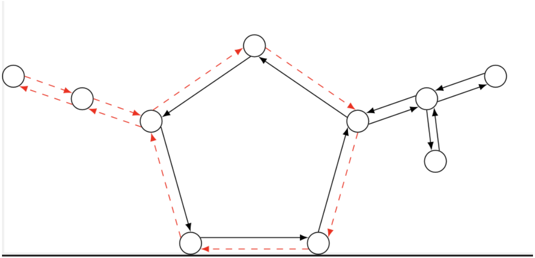

Figure 7: Double (opposite) unicyclic component.

- l single edges in $G_{k_1}, G_{k_2} \implies \mathbb{E}[a_{\psi_1}] = \mathbb{E}[a_{\psi_1}] = 0$.
- $a_t(1,2) = (\mathbb{E}[a_{1,2}a_{2,1}])^{\frac{k_1+k_2}{2}} = t^{\frac{k_1+k_2}{2}}$.

Parallel: $\mathbb{E}[a_{1,2}^2] = 0$.

- $a_t(1,2) \neq 0 \implies l = 1$.
- $a_t(1,2) = \mathbb{E}[a_{1,1}^2] t^{\frac{k_1+k_2}{2}-1} = t^{\frac{k_1+k_2}{2}}$.

Counting partitions

Lemma (From [MMP21, Lemma 39], [Min19], [MST09])

$$Card(\mathcal{P}_l(k_1, k_2)) = nc^{(l)}(k_1, k_2).$$

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Proposition (Diagonal covariance for Chebyshev polynomials)

$$\varphi^{(t)}(P_k^{(t)}, P_l^{(t)}) = kt^k \delta_{k=l}.$$

$$\varphi_c^{(t)}(P_k^{(t)}, P_l^{(t)}) = k\delta_{k=l}.$$

Relation with known results

$t = 1$:

- $A_{n,t} \sim$ GUE, $P_k^{(1)} = P_k$ usual Chebyshev.
- $\varphi^{(1)}(X^{k_1}, X^{k_2}) = \varphi_c^{(1)}(X^{k_1}, X^{k_2}) = nc(k_1, k_2)$.
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$t = 0$:

- $A_{n,t} \sim$ Ginibre, $P_k^{(0)} = X^k$.
- $\varphi^{(0)}(P_k^{(0)}, P_l^{(0)}) = 0$, $\varphi_c^{(0)}(P_k^{(0)}, P_l^{(0)}) = k\delta_{k=l}$.
- $\{Z_{k,0}, k \geq 0\}$ indep, $\mathbb{E}[Z_{k,0}^2] = 0$, $\mathbb{E}[|Z_{k,0}|^2] = k$.

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$$\{P_k^{(t)}\}_{k\geq 1} \equiv \text{ interpolation of diagonalising families.}$$

Conclusion

Theorem

Let $t \in [0,1]$. As $n \to \infty$,

$$f_{n,t} \underset{law}{\to} f_t : z \mapsto \kappa_t(z) \exp\left(-\sum_{k \geq 1} X_k^{(t)} \frac{z^k}{\sqrt{k}} \right)$$

with $\{X_k^{(t)}\}_{k \geq 1}$ independent complex Gaussian variables,

$$\mathbb{E}\left(X_k^{(t)}\right)^2 = t^k, \quad \mathbb{E}|X_k^{(t)}|^2 = 1$$

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Corollary (No Outlier)

$$C \cap E_t = \emptyset \implies N_n(C) := \#\left\{1 \leq i \leq n : \frac{\lambda_i}{\sqrt{n}} \in C\right\} \underset{\mathbb{P}}{\to} 0.$$

Future perspectives

- Case of $\{0,1\}$ matrices.
- Tightness for Coulomb gases.
- Universality : elliptic matrices & optimal moment conditions.

Thank you for your attention !

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